

# Charged Quantum Black Holes : Thermal Stability Criterion

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## Abstract

A criterion of thermal stability is derived for electrically charged *quantum* black holes having large horizon area (compared to Planck area), as an inequality between the mass of the black hole and its microcanonical entropy. The derivation is based on key results of Loop Quantum Gravity and equilibrium statistical mechanics of a grand canonical ensemble, with Gaussian fluctuations around an equilibrium thermal configuration assumed here to be a quantum *isolated* horizon. No aspect of classical black hole geometry is used to deduce the stability criterion. Since, no particular form of the mass function is used *a priori*, our stability criterion provides a platform to test the thermal stability of a black hole with a given mass function. The mass functions of the two most familiar charged black hole solutions are tested as a fiducial check. We also discuss the validity of the saddle point approximation used to incorporate thermal fluctuations. Moreover, the equilibrium Hawking temperature is shown to have an additional quantum correction over the semiclassical value.

## 1 Introduction

The idea of a *quantum* black hole has been articulated longer than a decade ago within Loop Quantum Gravity [1], [2], [3], [4]. This is an effective description wherein a three dimensional  $SU(2)$  Chern Simons Theory [1], [5], [6] coupled to point-like sources on punctures made by edges of spin networks describing bulk quantum geometry, is shown to describe a quantum *isolated* horizon [7], [9] or, in other words, a *null trapping horizon* [11]. The dimensionality of the Hilbert space of the Chern Simons theory leads, for large horizon areas (in comparison to the Planck area) to the Bekenstein-Hawking area law together with universal quantum corrections [3]. These results have been rigorously reviewed and rederived recently by a number of authors [12], [13], [14].

Quantum black holes not isolated from an ambient thermal reservoir have also been considered in the past [15], [16–18], [19]. In this approach one uses certain key results of Loop Quantum Gravity like the discrete spectrum of the area operator [27, 28] and the central assumption that the thermal equilibrium configuration is indeed an isolated horizon whose microcanonical entropy, including quantum spacetime fluctuations have already been computed via Loop Quantum Gravity. The idea here has been the study of the interplay of thermal and quantum fluctuations, and a criterion for thermal stability of such horizons has been obtained [18], [19, 20], using a ‘thermal holographic’ description involving a canonical ensemble and incorporating Gaussian thermal fluctuations. The generalization to horizons carrying charge has also been attempted, using a grand canonical ensemble, even though a somewhat ad hoc mass spectrum has been assumed [17].

In this paper, we attempt to rederive a thermal stability criterion for charged quantum horizons, *without* any ad hoc assumptions on the mass spectrum. With the benefit of hindsight, arguments which place the earlier formulation on a more solid footing are presented, together with novel aspects which enable us to sidestep earlier restrictions. A comparison with semiclassical thermal stability analyses of black holes [22] is made wherever possible. The range of validity of the saddle point approximation around the equilibrium configuration is examined to ensure the self-consistency of the Gaussian approximation.

The paper is organized as follows: In Subsection(2.1) there is a short account on the mass associated with the horizon of a black hole. In the rest of Section(2) we review the idea of Thermal Holography as laid out

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in [20]; the primacy of the boundary partition function of a Grand Canonical ensemble, in situations where the bulk Hamiltonian is a constraint, is established. The boundary partition function is then evaluated in Section(3) within the saddle point approximation, choosing an isolated horizon as the equilibrium configuration. Valid existence of the saddle point is shown to lead to the thermal stability criteria in terms of a second order partial differential inequalities involving the equilibrium mass and the microcanonical entropy. Solving one of those inequalities yields a more accessible form of the stability criterion in terms of a competition between ‘energy-driven’ and ‘entropy-driven’ aspects of the system. In Section(4) the range of validity of the saddle point approximation for the different cases is discussed. thermal stability criterion with semiclassical thermal stability properties of Reissner-Nordstrom and Anti-de Sitter Reissner-Nordstrom black holes is given. Section(5) contains our concluding remarks.

## 2 Thermal Holography

### 2.1 Horizon Energy

For a consistent Hamiltonian evolution for spacetimes admitting internal boundaries (isolated horizons, representing black holes at equilibrium) there must be a first law associated with each internal boundary( $b$ ), assumed to be a null hypersurface with the properties of a ‘one-way membrane’ [7,9] given by

$$\delta E_b^t = \frac{\kappa^t}{8\pi} \delta A_b + \Phi^t \delta Q_b$$

where  $E_b^t$  is the classical energy function associated with the horizon,  $\kappa^t$  and  $\Phi^t$  are the surface gravity and the electric potential respectively of the horizon,  $Q_b$  is the horizon charge. All the quantities are defined for a particular choice of time evolution vector field  $t^\mu$ . The family of time evolution vector fields  $[t^\mu]$  satisfying such first laws on the horizon are the permissible time evolution vector fields. These evolution vector fields also need to satisfy other boundary conditions. Each of these time evolution vector fields associates a classical energy function with the horizon which is a function of area and charge for Einstein-Maxwell theory. In arbitrary non-stationary cases (radiation may be present arbitrarily close to the horizon) for a particular time evolution vector field ( $t$ ), the Hamiltonian formulation yields

$$H^t = E_{ADM}^t - M_b^t$$

where  $H^t$  = Hamiltonian associated with the spacetime region in between the black hole boundary( $b$ ) and the boundaries at infinity,  $M_b^t$  = the mass associated with the horizon ( $b$ ),  $E_{ADM}^t$  = the usual ADM energy associated with the spatial boundary at infinity for a permissible vector field  $t^\mu$ .  $H^t$  is the Hamiltonian of the covariant phase space, which is the space of various class of solutions of the Einstein equations admitting internal boundaries. For stationary spacetimes the global timelike Killing field ( $\xi^\mu$ ) is the time evolution vector field. On physical ground one can say that there is nothing between the internal boundary and the boundary at infinity for stationary spacetimes, hence  $H^\xi = 0$ . On mathematical ground one can argue that in the Hamiltonian framework, for the stationary black hole solutions, the total Hamiltonian function  $H^\xi$  (which generates evolution along  $\xi^\mu$ ), must vanish as a first class constraint on the phase space [8,9]. This gives  $M_b^\xi = E_{ADM}^\xi$ . This is exactly what has been proved in another manner in the literature : that in stationary black hole spacetimes the ADM mass equals the energy of the black hole. Hence it is logical to identify  $E_b^\xi$  with the horizon mass  $M_b$  in the stationary case. The difference for an arbitrary non-stationary case is that  $H^t \neq 0$ . Thus it can be called as the mass associated with the Isolated Horizon in an active sense, that can change from one dynamic equilibrium situation to another satisfying the first law. Here, one should be careful that this mass associated with the isolated horizon is completely physical and is not to be confused with the Hamiltonian of the  $SU(2)$  Chern-Simons theory which describes horizon field degrees of freedom. The Hamiltonian for Chern Simons theory vanishes identically, since the theory is topological and insensitive to small metric deformations.

Clearly, the horizon mass is *not* affected by boundary conditions at asymptopia. It is defined *locally* on the horizon without referring to the asymptotic structure at all. The asymptotic conditions only modify the energy associated with the boundary at infinity and the bulk equation of motion (Einstein equations) [9,10]. The Hamiltonian framework discussed above is equally applicable for asymptotically flat and AdS spacetimes.

### 2.2 Quantum Geometry

For a classical spacetime with boundary, boundary conditions determine the boundary degrees of freedom and their dynamics. For a quantum spacetime, on the other hand, fluctuations of the boundary degrees of freedom

have a ‘life’ of their own (see for instance ref. [1]). Consequently, the Hilbert space of a quantum spacetime with boundary has the tensor product structure  $\mathcal{H} = \mathcal{H}_v \otimes \mathcal{H}_b$ , with the subscript  $v$  ( $b$ ) denoting the bulk (boundary) component. Thus, any generic state in quantum geometry,  $|\Psi\rangle$ , admits the expansion

$$|\Psi\rangle = \sum_{v,b} C_{vb} |\psi_v\rangle \otimes |\chi_b\rangle. \quad (2.2.1)$$

In presence of electromagnetic fields, one can consider  $|\psi_v\rangle$  (resp.  $|\chi_b\rangle$ ) to be the composite quantum gravity + quantum electrodynamics bulk (resp. boundary) state. The bulk states are annihilated by the *full* bulk Hamiltonian :  $\hat{H}_v |\psi_v\rangle \equiv [\hat{H}_{g,v} + \hat{H}_{e,v}] |\psi_v\rangle = 0$ ; this is the quantum version of the classical Hamiltonian constraint [28]. The total Hamiltonian operator acting on the generic state  $|\Psi\rangle$  has the form

$$\hat{H}_T |\Psi\rangle = (\hat{H}_v \otimes I_b + I_v \otimes \hat{H}_b) |\Psi\rangle \quad (2.2.2)$$

where,  $I_v(I_b)$  corresponds to the identity operator on  $\mathcal{H}_v(\mathcal{H}_b)$ .

While defining the grand canonical partition function, the charge operator ( $\hat{Q}$ ) for the black hole is also needed. It can be written in a similar fashion like the Hamiltonian as

$$\hat{Q} |\Psi\rangle = (\hat{Q}_v \otimes \hat{I}_b + \hat{I}_v \otimes \hat{Q}_b) |\Psi\rangle$$

where  $\hat{Q}_v$  and  $\hat{Q}_b$  are corresponding charge operators for the bulk states  $|\psi_v\rangle$  and the boundary states  $|\chi_b\rangle$ , respectively. In the classical theory the charge of a black hole is defined on the horizon i.e the internal boundary of the four dimensional spacetime (e.g. one can see how charge can be properly defined for spacetimes admitting internal boundaries in Einstein-Maxwell or Einstein-Yang-Mills theories in [7]). There is *no* charge associated with the bulk black hole spacetime, i.e.  $Q_v \approx 0$ , which is basically the Gauss law constraint for electrodynamics. Hence, its quantum version is *assumed* to be

$$\hat{Q}_v |\psi_v\rangle = 0.$$

Combining the quantum constraints on the Hamiltonian and charge operators we can define a new quantum constraint as

$$\hat{H}'_v |\psi_v\rangle = 0$$

where  $\hat{H}'_v \equiv \hat{H}_{T_v} - \Phi \hat{Q}_v$  and  $\Phi$  may be any function. But in our case it is a physically significant quantity which will be defined in the next paragraph. The implications of these quantum constraints will be seen during the construction of the grand canonical partition function.

### 2.3 The Partition Function

Let us consider a grand canonical ensemble of massive charged black holes immersed in a heat bath at some finite temperature with which it can exchange energy and charge as well. We construct a partition function for the thermodynamic system. Using the usual definition of the grand canonical partition function we write

$$Z_G = \text{Tr} \exp -\beta \hat{H}_T + \beta \Phi \hat{Q}$$

where the trace is taken over all states.  $\Phi$  is the electrostatic potential and  $\hat{Q}$  is the charge operator for the black hole. To write it in the explicit form first we write down a general quantum state of the black hole as follows

$$|\Psi\rangle = \sum_{v,b} c_{vb} |\psi_v\rangle \otimes |\chi_b\rangle$$

Now, we can write the partition function as

$$\begin{aligned} Z_G &= \sum_{v,b} |c_{vb}|^2 \langle \chi_b | \otimes \langle \psi_v | \exp -\beta \hat{H}_T + \beta \Phi \hat{Q} | \psi_v \rangle \otimes | \chi_b \rangle \\ &= \sum_{v,b} |c_{vb}|^2 \langle \chi_b | \otimes \langle \psi_v | \exp -\beta \hat{H}'_v | \psi_v \rangle \otimes | \chi_b \rangle \end{aligned} \quad (2.3.1)$$

where,  $\hat{H}' = \hat{H}_T - \Phi \hat{Q}$ . Writing the new operator  $H'$  as  $(\hat{H}'_v \otimes \hat{I}_b + \hat{I}_v \otimes \hat{H}'_b)$  and using  $\hat{H}'_v |\psi_v\rangle = 0$ , the partition function comes out to be equal to the boundary partition function only i.e.  $Z_G = Z_{Gb}$ , where  $Z_{Gb}$  is the boundary partition function for the charged isolated horizon, given by

$$Z_{Gb} = \text{Tr}_b \exp -\beta (\hat{H}_b - \Phi \hat{Q}_b)$$

where it is assumed that the boundary states can be normalized through the squared norm  $\sum_v |c_{vb}|^2 \langle \psi_v | \psi_v \rangle = |C_b|^2$ . This is analogous to the canonical ensemble scenario described in [19].

Now, the spectrum of the boundary Hamiltonian operator is still unknown in Loop Quantum Gravity. So we *assume* that the spectrum of the boundary Hamiltonian operator is a function of the discrete area spectrum and the charge spectrum of the area and the charge operators associated with the horizon, respectively<sup>1</sup>. The charge spectrum is equispaced on general physics grounds of charge quantization. The area spectrum, in Loop Quantum Gravity, can be *approximately* taken to be equispaced for large area black holes. For large area black holes, it is a well known fact in literature [31] that the major contribution to the entropy comes from the lowermost spins. Of course the higher spins contribute, but their contribution is exponentially suppressed. Thus, the effect of the corrections due to the inclusion of these higher spins is much more a technical aspect rather than physical. Hence, only spin 1/2 contributions for all punctures is taken into account, which leads to the equispaced area spectrum as an *approximation*, linear in the number of punctures.

In a basis in which both area and charge operators are diagonal, the partition function can be written as

$$Z_G = \sum_{k,l} g(k,l) \exp -\beta [E(A_k, Q_l) - \Phi Q_l] \quad (2.3.2)$$

where  $g(k,l)$  is the degeneracy corresponding to the area eigenvalue  $A_k$  and the charge eigenvalue  $Q_l$ .  $k, l$  are the area and charge quantum numbers respectively. As we are interested in regime of the large area and charge eigenvalues ( $k \gg 1, l \gg 1$ ), Poisson resummation formula is applied [17] to approximate the summation to an integration given by

$$Z_G = \int dx dy \exp -\beta \{E[A(x), Q(y)] - \Phi Q(y)\} g[A(x), Q(y)] \quad (2.3.3)$$

where  $x$  and  $y$  are area and charge quantum numbers in the continuum limit of  $k$  and  $l$  respectively. Since  $A = A(x)$  and  $Q = Q(y)$ , we can write  $dx = \frac{dA}{A_x}$  and  $dy = \frac{dQ}{Q_y}$  to write the partition function in terms of area and charge as free variables as follows

$$\begin{aligned} Z_G &= \int \frac{dA}{A_x} \frac{dQ}{Q_y} g(A, Q) \exp -\beta \{E(A, Q) - \Phi Q\} \\ &\approx \int dA dQ e^{S(A) - \beta E(A, Q) + \beta \Phi Q} \end{aligned} \quad (2.3.4)$$

where  $\frac{g(A, Q)}{A_x Q_y} = e^{S(A)}$  [25].  $S(A)$  being the microcanonical entropy is a function of black hole area alone, as has been established within loop quantum gravity [1, 3, 4].

### 3 Stability Against Gaussian Thermal Fluctuations

#### 3.1 Saddle Point Approximation(S.P.A.)

Having a well defined partition function, we investigate its finiteness under Gaussian thermal fluctuations about stable equilibrium configurations of the black hole given by the saddle points  $\{\bar{A}, \bar{Q}\}$ . Taylor expanding  $(S(A) - \beta E(A, Q) + \beta \Phi Q)$  about a saddle point  $(\bar{A}, \bar{Q})$  and applying the saddle point conditions one can rewrite the partition function as

$$\begin{aligned} Z_G &= e^{[S(\bar{A}) - \beta M(\bar{A}, \bar{Q}) + \beta \Phi \bar{Q}]} \\ &\times \int e^{-\frac{1}{2}[-\{S_{AA}(\bar{A}) - \beta M_{AA}(\bar{A}, \bar{Q})\}a^2 + \beta M_{QQ}(\bar{A}, \bar{Q})q^2 + 2\beta M_{AQ}(\bar{A}, \bar{Q})aq]} da dq \end{aligned} \quad (3.1.1)$$

where, we have used  $M(\bar{A}, \bar{Q})$  to indicate the equilibrium isolated horizon mass as a function of the area and charge. The saddle point conditions imply that the coefficients of  $a = (A - \bar{A})$  and  $q = (Q - \bar{Q})$  must vanish, which yield

$$\beta(\bar{A}, \bar{Q}) = \frac{S_A(\bar{A})}{M_A(\bar{A}, \bar{Q})}, \quad \Phi(\bar{A}, \bar{Q}) = M_Q(\bar{A}, \bar{Q}) \quad (3.1.2)$$

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<sup>1</sup>Actually this second assumption follows from the discussion in Subsection(2.1) [7, 9] for spacetimes admitting weakly isolated horizons where there exists a mass function determined by the area and charge associated with the horizon. This is an extension of that assumption to the quantum domain.

### 3.2 Quantum Surface Gravity

An interesting result of this statistical mechanical approach is that, it gives rise to a quantum correction to the surface gravity which is a direct consequence of the logarithmic corrections of the microcanonical entropy  $S = \frac{\bar{A}}{4} - \frac{3}{2} \log \frac{\bar{A}}{4}$  from loop quantum gravity. If one calculates  $\beta$  from the saddle point conditions and use it to find the quantum surface gravity ( $\kappa_{quantum}$ ) in terms of classical surface gravity ( $\kappa_{classical}$ ), there appear additional correction terms. One does this by calculating  $\beta$  from (3.1.2) and then using it in the expression  $\kappa_{quantum} = \frac{2\pi}{\beta}$  (more appropriately, this  $\beta$  can be replaced by  $\beta_{quantum}$ ). The quantum corrections to the classical surface gravity is found to be

$$\kappa_{quantum} \approx \kappa_{classical} \left( 1 + \frac{6}{\bar{A}} \right) \quad (3.2.1)$$

where higher order terms are neglected for large black holes ( $\bar{A} \gg 1$ ),  $\bar{A}$  being the area of the weakly isolated horizon in Planck units. One can easily check the formula by applying it to the Reissner-Nordstrom and AdS Reissner-Nordstrom cases. Since the formulation is not dependent on any particular situation (symmetry, etc.), it will also be valid for other massive charged black holes also.

### 3.3 Stability Criteria

For the integral (3.1.1) to be convergent, the Hessian matrix  $H$ , given by (3.3.1) has to be *positive definite* where

$$H = \begin{pmatrix} \beta M_{AA}(\bar{A}, \bar{Q}) - S_{AA}(\bar{A}) & \beta M_{AQ}(\bar{A}, \bar{Q}) \\ \beta M_{AQ}(\bar{A}, \bar{Q}) & \beta M_{QQ}(\bar{A}, \bar{Q}) \end{pmatrix} \quad (3.3.1)$$

The necessary and sufficient condition for the *real symmetric square matrix*  $H$  to be positive definite can be stated as follows [29] - ‘*Determinants of all the principal submatrices, including the determinant, of  $H$  are positive.*’ It is also a crucial point that the inverse temperature  $\beta$  must be positive. Hence, the necessary and sufficient conditions for the positive definiteness of the Hessian matrix lead to the following stability criteria

$$\beta \equiv \frac{S_A(\bar{A})}{M_A(\bar{A}, \bar{Q})} > 0 \quad (3.3.2)$$

$$\beta M_{AA}(\bar{A}, \bar{Q}) - S_{AA}(\bar{A}) > 0 \quad (3.3.3)$$

$$\det H \equiv \{ \beta M_{AA}(\bar{A}, \bar{Q}) - S_{AA}(\bar{A}) \} \beta M_{QQ}(\bar{A}, \bar{Q}) - \beta^2 M_{AQ}^2(\bar{A}, \bar{Q}) > 0 \quad (3.3.4)$$

It should be noted that, for  $M_{QQ}(\bar{A}, \bar{Q}) > 0$ , it will suffice to check only conditions (3.3.2) and (3.3.4) (which will be the case for RN and AdS RN black holes).

Here, one may wonder that how these stability criteria (3.3.2)–(3.3.4) are related to the convexity property of the entropy function, which is the usual notion for thermodynamic stability. It is true that the usual notion of thermodynamic stability is related to the convexity property of the entropy function. It is also true that this convexity property of the entropy function follows from the requirement of the convergence of the partition function under Gaussian thermal fluctuations [17, 22, 25]. Our stability criterion, likewise, follows from equations (3.3.2)–(3.3.4) which constitute the necessary and sufficient conditions for the grand canonical partition function to be well defined. Physically, the conditions to be satisfied by the entropy Hessian in [17], [22] and [25] (which imply convexity of entropy function) and the conditions (3.3.2)–(3.3.4) lead to the same conclusion i.e. the finiteness of the partition function under Gaussian thermal fluctuations. In fact, one can check that the apparent dissimilarity in the mathematical structure of the Hessian in [17] and eq. (3.3.1) is just a manifestation of the different variables which are summed over<sup>2</sup>. The equivalence between our conditions and the convexity of the entropy function is obvious for the charge-less case i.e. canonical ensemble, discussed in details in [19].

### 3.4 The Product Ansatz

The stability criterion (3.3.4) derived from the positive definiteness of the Hessian matrix, takes the form

$$\frac{M_{AA}}{M_A} - \frac{M_{QA}^2}{M_{QQ}M_A} > \frac{S_{AA}}{S_A} \quad (3.4.1)$$

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<sup>2</sup>To understand the motivation behind this choice of variables one can see Subsection (2.1) where a detailed explanation is given.

This is a second order partial differential inequality; one can resort to solving it for  $M(A, Q)$  in terms of  $S(A)$  and  $Q$  by trying to separate variables as one does for second order partial differential equations. Accordingly, we decompose  $M(A, Q) = \mu(A) \cdot \chi(Q)$  with  $\dim \mu = \dim M$  and  $\dim \chi = 0$ . Further,  $\chi(0) = 1$ , so as to agree with the result in the chargeless (Schwarzschild) case. The inequality (3.4.1) then reduces to

$$\frac{\mu}{\mu_A} \left[ \frac{\mu_{AA}}{\mu_A} - \frac{S_{AA}}{S_A} \right] > \frac{\chi_Q}{\chi} \frac{\chi_Q}{\chi_{QQ}} \quad (3.4.2)$$

Since the left and right sides of inequality (3.4.2) are functions of two completely independent variables  $\bar{A}$  and  $\bar{Q}$ , for the inequality to make sense, one may set the right side (say) to a constant  $1/\kappa$  say. One then obtains

$$\chi(\bar{Q}) = (1 + C\bar{Q})^{\frac{1}{\kappa-1}} \quad (3.4.3)$$

where,  $C$  is a constant with  $\dim C = \dim Q^{-1}$ . Substituting this in the inequality (3.4.2), we get

$$\mu(\bar{A}) > (\alpha S + \gamma)^{\frac{\kappa}{\kappa-1}} \quad (3.4.4)$$

where,  $\kappa > 1$ . Setting  $\gamma = 0$  for simplicity, and choosing  $(k_B \alpha)^{\frac{\kappa}{\kappa-1}} = M_P$ , we get

$$\begin{aligned} \frac{M}{M_P} &> \frac{S}{k_B} \left[ \frac{S}{k_B(1 + C\bar{Q})} \right]^{\frac{1}{\kappa-1}} \\ &= \frac{S}{k_B} \left[ 1 + \frac{1}{\kappa-1} \ln \left( \frac{S}{k_B(1 + C\bar{Q})} \right) + \dots \right] \end{aligned} \quad (3.4.5)$$

where the second line follows from the first in the limit  $\kappa \gg 1$ . It is easy to see that this bound is consistent with the chargeless case. It has to be seen how this bound checks out when the classical mass spectrum as a function of the area and charge of RN and AdS-RN black holes are used. The mass bound (3.4.5) is rewritten as

$$M(\bar{A}, \bar{Q}) > S(\bar{A}) \left[ \frac{S(\bar{A})}{1 + (\frac{4\pi\bar{Q}^2}{l_C^2})^{1/2}} \right]^{1/\kappa-1} \quad (3.4.6)$$

where we have replaced the constant  $C$  in terms of  $l_C$ . The classical RN is given by the metric ( $G = k_B = c = \hbar = 1$ )

$$ds^2 = -(1 - \frac{2M}{r} + \frac{Q^2}{r^2})dt^2 + (1 - \frac{2M}{r} + \frac{Q^2}{r^2})^{-1}dr^2 + r^2 d\Omega^2 \quad (3.4.7)$$

This leads to the mass associated with the RN horizon as a function of the horizon area and charge:

$$M_{RN}(A, Q) = \left( \frac{A}{4\pi} \right)^{1/2} \left[ 1 + \frac{4\pi Q^2}{A} \right] \quad (3.4.8)$$

Clearly, the charge is bounded by above according to  $4\pi Q^2 \leq A$ . Inserting this in equation (3.4.6) leads to

$$M(\bar{A}) > S \left[ \frac{S}{1 + S^{1/2}} \right]^{1/\kappa-1} \quad (3.4.9)$$

where, we have used  $S(A) = A/4l_P^2$  and set  $l_C = 2l_P$  for simplicity. We have reverted to the use of the Planck length  $l_P = (G\hbar/c^3)^{1/2}$  as a fundamental length scale signifying onset of LQG. For black holes of very large horizon area  $A \gg l_P^2$ , the above bound reduces to

$$M(\bar{A}) > S^{\frac{2\kappa-1}{2\kappa-2}} \quad (3.4.10)$$

In the same limit the classical  $M_{RN}$  obeys the upper bound  $M_{RN} < 2S^{1/2}$  which obviously violates the lower bound (3.4.10).

A similar analysis for the AdS-RN shows that for black holes of very large area compared to the cosmic area  $A_\Lambda \equiv 4\pi \left(-\frac{3}{\Lambda}\right) = 4\pi l^2$ , the classical black hole mass obeys the bound (3.4.10). This observation comes from the fact that the metric for AdS-RN is

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2\right)^{-1}dr^2 + r^2d\Omega^2 \quad (3.4.11)$$

where,  $\frac{\Lambda}{3} = -\frac{1}{l^2}$ . The mass associated with the AdS RN horizon is given by

$$M_{ADSRN}(A, Q) = \left(\frac{A}{4\pi}\right)^{1/2} \left[1 + \frac{4\pi Q^2}{A} + \frac{A}{4\pi l^2}\right] \quad (3.4.12)$$

It is obvious that for  $A \gg A_\Lambda$ , the last term in (3.4.12) dominates over the other two, and in this case, the bound is indeed obeyed. In fact the charge term makes a small contribution in any event, being bounded from above by 1.

**First Law :** The mass formula written in the form  $M(A, Q) = \mu(A)\chi(Q)$  satisfies the first law. Differentiating  $M$  we have

$$dM = \mu_A \chi dA + \mu \chi_Q dQ$$

Now, from (3.1.2) we have  $\Phi = M_Q = \mu \chi_Q$ . Hence

$$dM = \mu_A \chi dA + \Phi dQ$$

Again, finding  $\beta$  from (3.1.2) and using  $\kappa_{sg} = \frac{2\pi}{\beta}$  one finds that the surface gravity comes out to be  $\kappa_{sg} = 8\pi\mu_A\chi$  ('sg' stands for surface gravity). Then, it is easy to see that this  $\kappa_{sg}$  proves that the product ansatz satisfies the first law

$$dM = \frac{\kappa_{sg}}{8\pi} dA + \Phi dQ$$

Thus, from the thermodynamic point of view, this kind of solution for mass is acceptable.

### 3.5 The Classical Metrics

**Reissner-Nordstrom black hole :** In this paragraph we investigate the stability of massive charged Reissner-Nordstrom black holes against Gaussian thermal fluctuations. The classical Reissner-Nordstrom metric which is given by (3.4.7). The mass of the black hole in terms of area and charge as independent variables and it is given by

$$M = \frac{1}{2} \left(\frac{A}{4\pi}\right)^{\frac{1}{2}} (1 + \rho) \quad (3.5.1)$$

where  $\rho = \frac{4\pi Q^2}{A}$  is a dimensionless parameter in the chosen units. The microcanonical entropy including the logarithmic correction term from loop quantum gravity is given by

$$S = \frac{A}{4} - \frac{3}{2} \log \frac{A}{4} \quad (3.5.2)$$

Using (3.5.1) and (3.5.2) one finds that

$$\beta = 2\sqrt{\pi} A^{\frac{1}{2}} \frac{(1 - \frac{6}{A})}{(1 - \bar{\rho})} \quad (3.5.3)$$

$$\det H = -\frac{\pi}{2A} \frac{(1 - \frac{6}{A})(1 + \frac{6}{A})}{(1 - \bar{\rho})} \quad (3.5.4)$$

where  $\bar{\rho} = \frac{4\pi Q^2}{A}$ . Conditions (3.3.2) and (3.5.3) together imply  $\bar{\rho} < 1$ , whereas, (3.3.3) and (3.5.4) imply  $\bar{\rho} > 1$ . From this contradiction one can conclude that the Reissner-Nordstrom black hole is *locally unstable* against Gaussian thermal fluctuations.<sup>3</sup>

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<sup>3</sup>This result is in agreement with what has been told in [24] i.e. *an asymptotically flat, non-extremal black hole can never achieve a state of thermal equilibrium*. This paper also uses a statistical mechanical formulation without any classical metric but do not have a sound mathematical basis as a justification of the quantum statistical formulation which has been presented in our work.



**AdS Reissner-Nordstrom black hole :** In this paragraph we investigate the stability of massive charged AdS Reissner-Nordstrom black holes against thermal fluctuations. The classical AdS Reissner-Nordstrom metric which is given by (3.4.11). The mass of the AdS Reissner-Nordstrom black hole in terms of area and charge as the independent variables is given by

$$M = \frac{1}{2} \left( \frac{A}{4\pi} \right)^{\frac{1}{2}} (1 + \rho + \sigma) \quad (3.5.5)$$

where we have set  $\frac{\Lambda}{3} = -\frac{1}{l^2}$  as  $\Lambda$  is negative for AdS spacetimes and introduced the parameter  $\sigma = \frac{A}{4\pi l^2} = \frac{A}{A_\Lambda}$ . Here also the entropy is given by (3.5.2). Calculating the required quantities for AdS Reissner-Nordstrom case from (3.5.2) and (3.5.5) one finds that

$$\begin{aligned} \beta &= 2\sqrt{\pi}\bar{A}^{\frac{1}{2}} \frac{(1 - \frac{6}{\bar{A}})}{(1 - \bar{\rho} + 3\bar{\sigma})} \\ \det H &= \frac{\pi}{2\bar{A}} \frac{(1 - \frac{6}{\bar{A}})(1 + \frac{6}{\bar{A}})}{(1 - \bar{\rho} + 3\bar{\sigma})^2} \left[ \bar{\rho} - 1 + 3\bar{\sigma} \left( 1 - \frac{18}{\bar{A}} \right) \left( 1 + \frac{6}{\bar{A}} \right)^{-1} \right] \end{aligned}$$

where  $\bar{\sigma} = \frac{\bar{A}}{4\pi l^2} = \frac{\bar{A}}{A_\Lambda}$ . For the conditions (3.3.2) and (3.3.4) to be satisfied simultaneously the parameters have the following bound given by

$$1 - 3\bar{\sigma} \left( 1 - \frac{18}{\bar{A}} \right) \left( 1 + \frac{6}{\bar{A}} \right)^{-1} < \bar{\rho} < 1 + 3\bar{\sigma}$$

which in the first order approximation reduces to the bound

$$1 - 3\bar{\sigma} \left( 1 - \frac{24}{\bar{A}} \right) < \bar{\rho} < 1 + 3\bar{\sigma}$$

Hence, AdS Reissner-Nordstrom black holes are *locally stable* against thermal fluctuations for the above range of parameters.

## 4 Validity of S.P.A.

To ensure the validity of the saddle point approximation, the relative r.m.s. fluctuations about the saddle point are to be checked i.e.  $\Delta A_{rms}/\bar{A} = (\Delta A^2)^{1/2}/\bar{A}$ ,  $\Delta Q_{rms}/\bar{Q} = (\Delta Q^2)^{1/2}/\bar{Q}$  where

$$\Delta A^2 = (H^{-1})_{11} = \frac{\beta M_{QQ}}{\det H} \quad , \quad \Delta Q^2 = (H^{-1})_{22} = \frac{\beta M_{AA} - S_{AA}}{\det H}$$

### 4.1 The Product Ansatz

Using the product ansatz the mean square fluctuation of the charge comes out to be

$$\Delta Q^2 = \frac{(S_A \mu_{AA} - S_{AA} \mu_A) \mu_A \chi^2}{S_A (S_A \mu_{AA} - S_{AA} \mu_A) \mu \chi_{QQ} \chi - S_A^2 \mu_A^2 \chi_Q^2} \quad (4.1.1)$$

Now, the following quantities are calculated

$$\begin{aligned} 1 + CQ &= \left( \frac{\kappa - 1}{\mu C} \right)^{\frac{\kappa-1}{2-\kappa}} \Phi^{\frac{\kappa-1}{2-\kappa}} \\ \chi &= (1 + CQ)^{\frac{1}{1-\kappa}} = \left( \frac{\kappa - 1}{\mu C} \right)^{\frac{1}{2-\kappa}} \Phi^{\frac{1}{2-\kappa}} \\ \chi_Q &= \frac{C}{\kappa - 1} (1 + CQ)^{\frac{2-\kappa}{\kappa-1}} = \frac{\Phi}{\mu} \\ \chi_{QQ} &= \frac{C^2(2-\kappa)}{(\kappa-1)^2} (1 + CQ)^{\frac{3-2\kappa}{\kappa-1}} = \frac{C^2(2-\kappa)}{(\kappa-1)^2} \left( \frac{\kappa-1}{\mu C} \right)^{\frac{3-2\kappa}{2-\kappa}} \Phi^{\frac{3-2\kappa}{2-\kappa}} \end{aligned} \quad (4.1.2)$$



Using all these quantities in (4.1.1) one finds that

$$\Delta Q^2 = \frac{\eta/S_A}{\eta\mu(2-\kappa) - S_A\mu_A^2} \quad (4.1.3)$$

where  $\eta = S_A\mu_{AA} - S_{AA}\mu_A$ . Now, condition (3.3.4) implies  $M_{QQ}(\beta M_{AA} - S_{AA}) > \beta M_{AQ}^2$ . Since the R.H.S. is positive definite ( $\beta > 0$ ), the L.H.S. has to be positive definite. This implies either  $M_{QQ} > 0$ ,  $(\beta M_{AA} - S_{AA}) > 0$  or  $M_{QQ} < 0$ ,  $(\beta M_{AA} - S_{AA}) < 0$ . The second one is not possible as it makes  $\Delta Q^2$  negative. Hence the only possibility is the first one. Since  $S_A > 0$ , hence  $(\beta M_{AA} - S_{AA}) > 0$  implies  $\eta > 0$ . Keeping all these conditions in mind if one goes back to the expression (4.1.3), one finds that for  $\Delta Q^2 > 0$ ,  $\kappa$  must satisfy the condition  $\kappa < 2$ . The relative r.m.s. fluctuation of the charge goes as

$$\frac{(\Delta Q)_{rms}}{Q} = \frac{4C\pi^{\frac{1}{4}} \left(1 + \frac{12}{A}\right)^{\frac{1}{2}}}{\left\{ \left(1 + \frac{6}{A}\right)(2-\kappa) + \left(1 - \frac{6}{A}\right) \right\}^{\frac{1}{2}} \left\{ \left(\frac{\kappa-1}{\mu C}\Phi\right)^{\frac{\kappa-1}{2-\kappa}} - 1 \right\}} \bar{A}^{-\frac{1}{4}} \quad (4.1.4)$$

where  $\mu$  and its derivatives are calculated from the relation  $A = 8\pi\mu^2$ . The argument is as follows. The irreducible mass is a function of area only and it is the Schwarzschild mass. Hence, for  $Q = 0$  ( $\chi(0) = 1$ ) we have  $A = 8\pi\mu^2$ . The mean square of area fluctuations comes out to be

$$\Delta A^2 = \frac{\mu\mu_A}{\eta\mu - \frac{S_A\mu_A^2}{2-\kappa}}$$

Thus the relative rms fluctuation of area comes out to be

$$\frac{(\Delta A)_{rms}}{\bar{A}} = \left( \frac{\mu\mu_A}{\eta\mu - \frac{S_A\mu_A^2}{2-\kappa}} \right)^{\frac{1}{2}} \frac{1}{16\pi\mu^2} \quad (4.1.5)$$

To see the variation of (4.1.5) with  $A$  it is written explicitly in terms of  $A$  as follows

$$\frac{(\Delta A)_{rms}}{\bar{A}} = \frac{2\sqrt{2}}{\left\{ \left(1 - \frac{6}{A}\right)(2-\kappa)^{-1} - \left(1 + \frac{6}{A}\right) \right\}^{\frac{1}{2}}} \bar{A}^{-\frac{1}{2}} \quad (4.1.6)$$

This again shows that the constant  $\kappa$  has an upper bound which is not much larger than unity. Moreover, calculation yields  $\eta = -\frac{1}{64\sqrt{\pi}}(1 + \frac{6}{A})\bar{A}^{-\frac{3}{2}}$  which is a negative definite quantity (contradicts  $\eta > 0$ ). For  $\eta < 0$  we must have  $\chi_{QQ} < 0$  which in turn implies  $\kappa > 2$ . But this step i.e.  $\eta < 0$  will lead to negative  $\Delta Q^2$ , which again implies instability. Hence  $\kappa$  has to be bounded from above by 2. Considering the appropriate bounds of  $\kappa$  it is seen from (4.1.4) and (4.1.6) that the relative r.m.s. fluctuations fall off for large area of the weakly isolated horizon i.e.  $\bar{A} \gg 1$  in Planck units.

## 4.2 The Classical Metrics

**Reissner-Nordstrom black hole :** The mean square fluctuations of area and charge in this case come out to be negative due to the presence of the determinant of the Hessian matrix. This emphasizes the instability of Reissner-Nordstrom black holes against Gaussian thermal fluctuations.

**AdS Reissner-Nordstrom black hole :** The relative r.m.s. fluctuations of area and charge in this case are found to be

$$\frac{\Delta A_{rms}}{\bar{A}} = \left[ \frac{8(1 - \bar{\rho} + 3\bar{\sigma})}{\left(1 + \frac{6}{A}\right) \left\{ \rho - 1 + 3\bar{\sigma} \left(1 - \frac{18}{A}\right) \left(1 + \frac{6}{A}\right)^{-1} \right\}} \right]^{\frac{1}{2}} \bar{A}^{-\frac{1}{2}} \quad (4.2.1)$$

$$\frac{\Delta Q_{rms}}{Q} = \frac{\sqrt{3}}{\Phi^2} \left[ \frac{(1 - \bar{\rho} + 3\bar{\sigma}) \left(1 - \frac{2}{A}\right) \left\{ \bar{\rho} + \bar{\sigma} \left(1 - \frac{18}{A}\right) \left(1 - \frac{2}{A}\right)^{-1} - \frac{1}{3} \left(1 + \frac{6}{A}\right) \left(1 - \frac{2}{A}\right)^{-1} \right\}}{\left(1 - \frac{6}{A}\right) \left(1 + \frac{6}{A}\right) \left\{ \rho - 1 + 3\bar{\sigma} \left(1 - \frac{18}{A}\right) \left(1 + \frac{6}{A}\right)^{-1} \right\}} \right]^{\frac{1}{2}} \bar{A}^{-\frac{1}{2}} \quad (4.2.2)$$

where the equilibrium charge has been replaced in terms of the electric potential  $\Phi$  which can be easily calculated from the saddle point conditions (3.1.2). It is clearly seen from the above expressions that the fluctuations fall off in the large area regime. As far as the positivity of the mean square fluctuations (r.m.s. fluctuation to be real) is concerned one needs to be more careful with the lower bound of  $\bar{\rho}$ . One can show that there arise the following two cases :

- 1) For  $\bar{\sigma} < \frac{1}{3} \left(1 + \frac{24}{A}\right)$ , the correct lower bound for  $\bar{\rho}$  is what has been obtained i.e.  $1 - 3\bar{\sigma} \left(1 - \frac{24}{A}\right)$ .
- 2) For  $\bar{\sigma} > \frac{1}{3} \left(1 + \frac{24}{A}\right)$ , the correct lower bound for  $\bar{\rho}$  is given by  $1 - 3\bar{\sigma} \left(1 - \frac{16}{A}\right) + \frac{8}{A}$ .

## 5 Discussion

The first thing to say about this approach is that its origin is purely based on quantum aspects of spacetime. During the build up of the formalism, *no classical metric* is used. The construction of the partition function is purely based on the ideas and results of *Loop Quantum Gravity* e.g. the use of Chern-Simons states, the splitting up of the total Hilbert space, etc. and also on the Hamiltonian formulation of spacetimes admitting weakly isolated horizons. The entropy correction also follows from the quantum theory. The classical metrics come to the picture only to be tested.

In course of this heuristic statistical mechanical approach of stability analysis of black holes, broadly two assumptions are made. In classical Hamiltonian GR it is known that the total Hamiltonian (gravity + matter) vanishes. So, it is very logical to consider that the quantum total Hamiltonian operator annihilates the bulk states of quantum matter coupled spacetime. A similar argument follows for the assumption of the quantum constraint on the volume charge operator. These two assumptions may be considered to be one due to their fundamental similarity and they ultimately give rise to a single quantum constraint.

In Section(2), a second assumption is made regarding the eigenvalue spectrum of the energy of the black hole. It is already mentioned (in a footnote) that this second assumption is not a strong one. If one studies [7] carefully, the classical mass function associated with the horizon is stipulated to be a function of horizon area and charge. Again, this horizon area and charge are the functions of the local fields on the horizon. Proper quantization of the classical horizon area and charge will obviously lead to a well defined boundary Hamiltonian operator. The fact that there exists a quantum boundary Hamiltonian operator which acts on the boundary Hilbert space of the black hole is an assumption, since the exact form of such a Hamiltonian operator is still unknown. But the fact that its eigenvalue spectrum is a function of eigenvalue spectra of the area and charge operators is most likely a valid assumption, as it is bound to happen if such a boundary Hamiltonian operator exists. It follows from the classical analog - the mass associated with the horizon must be a function of the horizon area and charge for a consistent Hamiltonian evolution [7].

In ref. [17] where a similar approach has been taken, a particular functional form of the mass in terms of the area and charge had been used on an ad hoc basis. Such an ad hoc assumption has been shown here to be quite redundant. This, therefore, is a significant strength of the paper, relative to the earlier assay. Thus, the statistical mechanical approach adopted in this paper, though similar, now stands on a far stronger ground than in the previous version.

This statistical mechanical approach gives us a new quantum correction to the surface gravity arising from the loop quantum gravity corrections to the microcanonical entropy. One can easily check its validity. Moreover, it predicts local thermodynamic instability of the Reissner-Nordstrom black hole and local thermodynamic stability of the AdS Reissner-Nordstrom black hole as was shown classically in [21]. As far as the relative charge fluctuations are concerned, in literature [23] there are problems for AdS RN black holes as it does not fall off for any condition. This problem has been solved in this paper (Section 4). Last but not the least, once more a word is worth mentioning that this formulation *does not* involve any classical metric. The whole thing depends on quantum aspects of spacetime. One can generalize this to study rotating and even charged-rotating black holes because there is no use of symmetry in the theory. We look forward to give those analyses in future.

There is a more crucial issue which can be of utmost importance to have a deeper understanding of black hole thermodynamics [30]. As far as the saddle-point approximation is concerned, the Euclidean path integral approach does look similar to our thermal holographic approach. But there is a crucial difference between the two approaches. The path integral approach, within the saddle point approximation, requires information of the full black hole spacetime (bulk and horizon) as given by the classical black hole metric solution chosen to be the saddle point. Thus one needs global information away from horizon. In the thermal holographic approach which we adopt, one needs only *local information* associated with the equilibrium isolated horizon geometry interpreted as an inner null boundary. Thus, no detailed knowledge of the full classical spacetime is needed. The mass of the isolated horizon is an *unspecified* function of the area and charge, and we never need to specify

this function to derive our results, except as fiducial checks [Subsection (3.5)] appropriate to given classical metrics. This insensitivity of our approach to an explicit classical black hole metric is a key feature of our work and can be taken to mean that our results are in a sense more general than those computed from the Euclidean path integral.

A further distinction is that, in contrast to the Euclidean path integral approach, where quantum fluctuations around a *classical* metric are considered, our saddle point is a *quantum* isolated horizon whose quantum states and their (non-perturbative) dynamics are described by a quantum Chern Simons theory. Consequently, the *equilibrium* entropy is the microcanonical entropy computed in earlier work (ref. [1–4]) based on Loop Quantum Gravity, and already has an infinite series of corrections (including those logarithmic in horizon area) beyond the Bekenstein-Hawking area law, incorporating quantum spacetime fluctuations. In this paper, *additional thermal fluctuations* (and their physical effects) are considered, over and above the quantum spacetime fluctuations already incorporated for the equilibrium configuration. Quantum and thermal fluctuations are thus, treated somewhat distinctly in our approach, and the result is an interesting interplay between them. In the Euclidean path integral, such distinctions are not as clear. It might be of future interest to see better how these two somewhat disparate approaches may be related.

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